Some Vertex-Transitive Integral Graphs

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Outline

Some Motivation

Which graphs have integral spectra? **Harary, Schwenk; 1974**

- An `integral graph` is a graph whose adjacency matrix has only eigenvalues that are integers.
- Some applications: quantum information processing, load balancing problem in multiprocessor interconnection networks

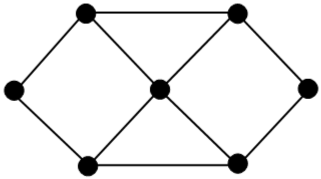
Ex// C_3, C_4, C_6, K_n, P_2, cube, triangular prism

Connected integral graphs with *n* vertices

n	1	2	3	4	5	6	7	8	9	10	11	12
#	1	1	1	2	3	6	7	22	24	83	236	325

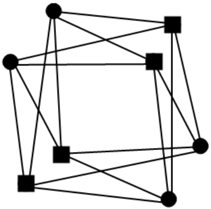
Initial Items

- G will be a simple, connected graph with n vertices.
- The `spectrum` of a graph, $Sp(G)$, is the eigenvalues with their multiplicity.



$$Sp(G) = \{3, 1^2, 0, -1, -2^2\}$$

Connected 4-Regular Bipartite Integral Graphs



$$Sp(G) = \{4, 0^6, -4\}$$

- Regular (or bounded degree) - there are finitely many
- 4-Regular - the largest eigenvalue is 4 with multiplicity 1
- Bipartite - eigenvalues are symmetric with respect to 0
- 4-Regular Bipartite - gives $n \leq 6560$ and
  $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -1^x, -4\}$

Lists of possible spectra

We let C_k denote the cycle on k vertices.

n	x	y	z	w	C_4	C_6
8	0	0	0	3	36	96
10	0	0	4	0	30	130
12	0	1	4	0	27	138
12	0	2	0	3	30	112
14	1	0	3	2	36	102
			⋮			
560	76	84	84	35	0	0

This list of possible spectra of 4-regular integral graphs can be found in a **2007** paper by Stevanovic, de Abreu, de Freitas, and Del-Vecchio.

Counting Cycles at Each Vertex

$$\#C_k \text{ per vertex} = \frac{k * (\#C_k)}{n} \in \mathbb{Z}^+$$

Ex// Given that $n = 48$,
$Sp(G) = \{4, 3^5, 2^6, 1^{11}, 0^2, -1^{11}, -2^6, -3^5, -4\}$, and that
$C_4 = 24$ and $C_6 = 140$;

$$\frac{k * (\#C_k)}{n} = \frac{4 * (24)}{48} = 2 \in \mathbb{Z}^+$$

$$\frac{k * (\#C_k)}{n} = \frac{6 * (140)}{48} = \frac{35}{2} \notin \mathbb{Z}^+$$

$\therefore$ The possible spectra list entry, 48 5 6 11 1 24 140, can't be realized by a vertex-transitive graph.

Counting Closed Walks

- For eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$;

$$\sum_{i=1}^{n} \lambda_i^j = \#\text{closed walks of length } j$$

- For an r-regular graph, we have a generating function for counting closed walks around various small subgraphs;

$$n(1+5r+9r^2+5r^3)+6C_3+48rC_4+24C_{3\cdot 3}+12\Theta_{2,2,1}+12C_6$$

counts closed walks of length 6.

Counting Closed Walks

The # of closed walks of length 8 in a bipartite graph G:

$$\sum_{i=1}^{n} \lambda_i^8 = 2092n + 2024[C_4] + 288[C_6] + 16[C_8] + 96[\Theta_{2,2,2,2}] + 48[\Theta_{2,2,2}] + 16[\Theta_{3,3,1}]$$

where $\Theta_{i_1,i_2,\ldots,i_h}$ consists of two vertices joined by internally disjoint paths of lengths i_j for $j = 1, \ldots, h$; and $[S]$ denotes the number of S subgraphs in G.

- Often $[C_8]$ could now be determined and then $\frac{8*[C_8]}{n}$ was used to eliminate possible spectra unable to realize a vertex-transitive graph.

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Elimination Results

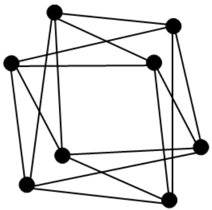
List of possible spectra - connected 4-regular integral bipartite:

- $8 \leq n \leq 560$
- 43 different values for n
- 828 different entries

List of possible spectra - connected 4-regular vertex-transitive integral bipartite (VT):

- $8 \leq n \leq 560$
- 29 different values for n
- 58 different entries

Cayley Graphs



$$Sp(G) = \{4, 1, 0^5, -1, -4\}$$

- This is the Cayley graph: $G(\{a, a^3, a^5, a^7\} : C_8)$
- For a group Γ and $S \subseteq \Gamma$, such that the identity element is not in S, the `Cayley graph` $G(S : \Gamma)$ has vertex set Γ and x adjacent to y when $xy^{-1} \in S$.

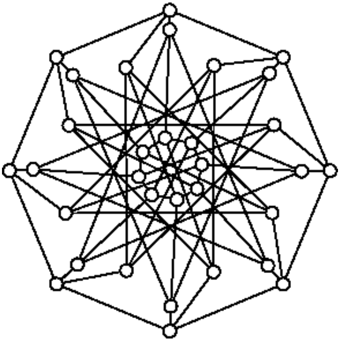
The Algorithm

- Consider all groups of order in the list of *VT* possible spectra.
- Form appropriate 4 element generating sets and build corresponding Cayley graphs.
- Take only those graphs that are connected and bipartite.
- Check if the eigenvalues are integers.

Quartic Bipartite Cayley Graphs We Found

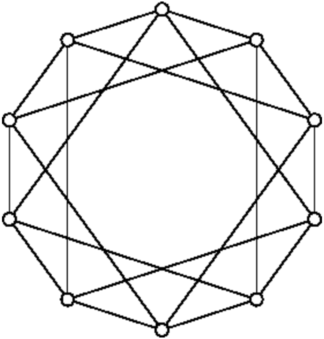
Cayley Integral Graphs

n	8	12	16	18	24	30	32	36	40	48	72	120
#	1	2	1	1	3	1	1	1	1	1	2	1



$$Sp(G) = \{4, 2^{12}, 0^6, -2^{12}, -4\}$$

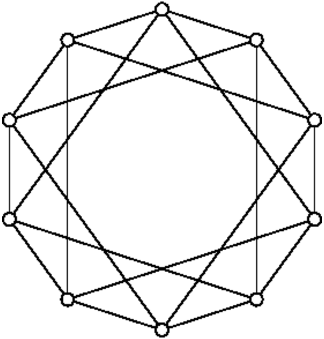
The Graph when $n = 10$



$$Sp(G) = \{4, 1^4, -1^4, -4\}$$

$$G(\{f, fr, fr^2, r^2f\} :< r, f \mid r^5, f^2, (rf)^2 >)$$
$$G(\{a, a^3, a^7, a^9\} :< a \mid a^{10} >)$$

The Graph when $n = 10$



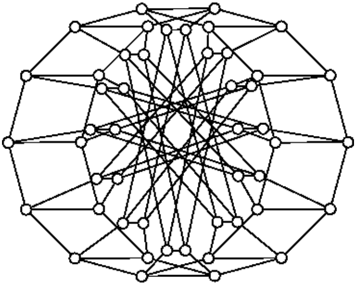
$$Sp(G) = \{4, 1^4, -1^4, -4\}$$

$G(\{S_0, S_1, S_2, S_3\} : D_{10})$
$G(\{a, a^3, a^7, a^9\} : C_{10})$

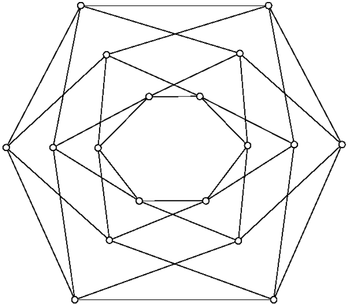
Isomorphic Cayley Graphs

The following property was used to eliminate not all but a great number of isomorphic graphs on the same group, Γ:

$$S^{\sigma} = T \text{ for some } \sigma \in Aut(\Gamma) \Rightarrow G(S : \Gamma) \cong G(T : \Gamma)$$



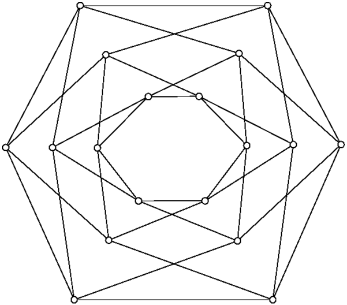
The Graph when $n = 18$



$$Sp(G) = \{4, 2^4, 1^4, -1^4, -2^4, -4\}$$

$G(\{c, ca, cb, cab\} :< a, b, c \mid a^3, b^3, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 >)$
$G(\{s, st, ats, a^2ts\} :< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >)$
$G(\{sa, sa^2, sat, sa^2t\} :< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >)$
$G(\{a, a^5, a^3b, a^3b^2\} :< a \mid a^6 > \times < b \mid b^3 >)$

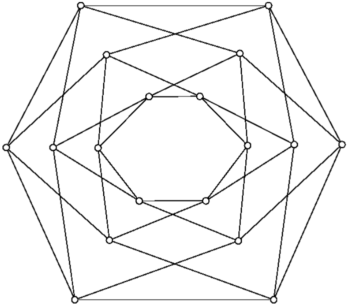
The Graph when $n = 18$



$$Sp(G) = \{4, 2^4, 1^4, -1^4, -2^4, -4\}$$

$G(\{c, ca, cb, cab\} : (C_3 \times C_3) \rtimes C_2)$
$G(\{s, st, ats, a^2ts\} : C_3 \times S_3)$
$G(\{sa, sa^2, sat, sa^2t\} : C_3 \times S_3)$
$G(\{a, a^5, a^3b, a^3b^2\} : C_6 \times C_3)$

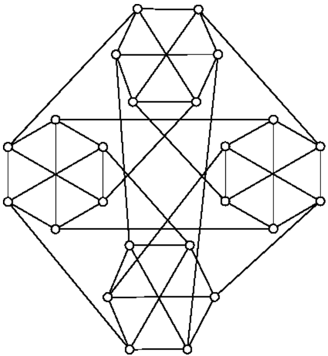
The Graph when $n = 18$



$$Sp(G) = \{4, 2^4, 1^4, -1^4, -2^4, -4\}$$

$G(\{c, ca, cb, cab\} : (C_3 \times C_3) \rtimes C_2)$
$G(\{s, st, ats, a^2ts\} : C_3 \times S_3)$ ——> 2 involutions
$G(\{sa, sa^2, sat, sa^2t\} : C_3 \times S_3)$ ——> 0 involutions
$G(\{a, a^5, a^3b, a^3b^2\} : C_6 \times C_3)$

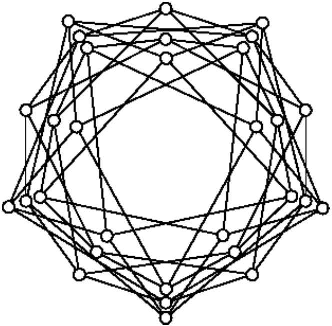
One of the Graphs when $n = 24$



$$Sp(G) = \{4, 3^3, 1^5, 0^6, -1^5, -3^3, -4\}$$

$$G(\{s, t^2st, st^2sts, stst^2st\} :< s, t \mid s^2, t^3, (st)^4 >)$$
$$G(\{a, as, at^2s, ast\} :< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >)$$

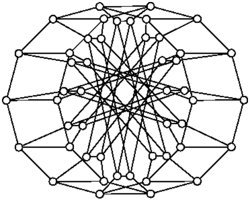
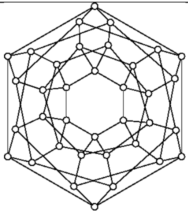
The Graph when $n = 30$



$$Sp(G) = \{4, 2^{10}, 1^{4}, -1^{4}, -2^{10}, -4\}$$

$$G(\{as, a^2st, a^4s, a^3st\} :< a \mid a^5 > \times < s, t \mid s^2, t^3, (st)^2 >)$$
$$G(\{f, r^2f, r^3f, r^{11}f\} :< r, f \mid r^{15}, f^2, (rf)^2 >)$$

Graphs and the Groups Responsible

 $Sp(G) = \{4, 3^6, 2^4, 1^{10}, 0^6, \ldots\}$	 $Sp(G) = \{4, 3^4, 2^4, 1^4, 0^{10}, \ldots\}$
$C_2 \times C_4 \times S_3$	$C_3 \times (C_3 \rtimes C_4)$
$D_8 \times S_3$	$(C_3 \times C_3) \rtimes C_4$
$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	$S_3 \times S_3$
$C_6 \times D_8$	$C_6 \times S_3$
$C_2 \times S_4$	$C_2 \times ((C_3 \times C_3) \rtimes C_2)$
$C_2 \times C_2 \times A_4$	$C_6 \times C_6$
$C_2 \times C_2 \times C_2 \times S_3$	

The End