Quasi m-Cayley strongly regular graphs

Klavdija Kutnar

University of Primorska, Koper, Slovenia

Feb, 2012

Joint work with Luis Martinez Fernandez, Aleksander Malnič and Dragan Marušič.

Strongly regular graphs

A regular graph X of valency k and order v is called strongly regular graph (SRG) with parameters (v, k, λ, μ) if any two adjacent vertices have λ common vertices and any two distinct non-adjacent vertices have μ common vertices.

Examples

For a prime power q such that $q \equiv 1 \pmod 4$ the Paley graph $P(q)$ is a graph with vertex set $\mathbb{F}_q$ in which two vertices are adjacent if their difference is non zero square.

$P(q)$ is a strongly regular graph with parameters

$$(v, k, \lambda, \mu) \; = \; (q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}).$$

m-Cayley graphs and semiregular automorphisms

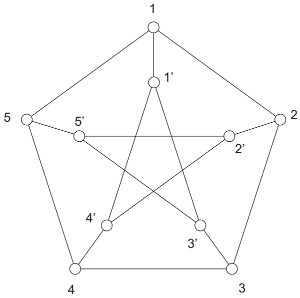
A graph X is an m-Cayley graph on a group H if its automorphism group admits a semiregular subgroup H having m orbits, all of equal length.

If H is cyclic and

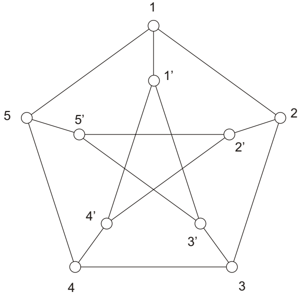
- $m = 1$ then X is said to be circulant;
- $m = 2$ then X is said to be bicirculant;
- $m = 3$ then X is said to be tricirculant.

A non-identity automorphism of a graph with m cycles of equal length n in its cycle decomposition is said to be (m, n)-semiregular.

Example

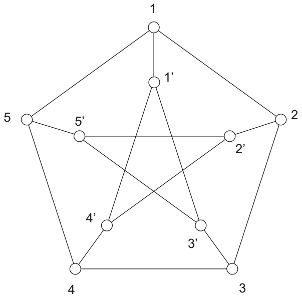


Example

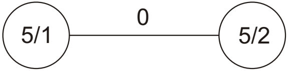


a $(2,5)$-semiregular automorphism $\rho = (1\,2\,3\,4\,5)(1'\,2'\,3'\,4'\,5')$

Example



a $(2,5)$-semiregular automorphism $\rho = (1\,2\,3\,4\,5)(1'\,2'\,3'\,4'\,5')$



A particular attention has been given to questions regarding strong regularity for various classes of graphs satisfying certain special symmetry conditions.

strongly regular Cayley graphs, strongly regular 2-Cayley graphs, strongly regular 3-Cayley graphs (Bridges, Mena, Leung, Ma, Marušič, Miklavič, Šparl, de Resmini, Jungnickel, ...)

For example, by a classical result of Bridges and Mena (1979) it is known that the Paley graphs are the only SRGs among Cayley graphs on cyclic groups.

Quasi m-Cayley graphs – quasi-semiregularity

A group G acts quasi-semiregularly on a set V if there exists an element ∞ in V such that the stabilizer G_∞ of the element ∞ in G is equal to G, and the stabilizer G_v of any element $v \in V - \{\infty\}$ in G is trivial. The element ∞ is called the point at infinity.

A group G acts quasi-semiregularly on a set V if there exists an element ∞ in V such that the stabilizer G_∞ of the element ∞ in G is equal to G, and the stabilizer G_v of any element $v \in V - \{\infty\}$ in G is trivial. The element ∞ is called the point at infinity.

A graph X is a quasi m-Cayley graph on a group H if its automorphism group admits a semiregular subgroup H having m orbits, all of equal length.

If H is cyclic and

- $m = 1$ then X is said to be quasi circulant;
- $m = 2$ then X is said to be quasi bicirculant;
- $m = 3$ then X is said to be quasi tricirculant;
- ...

Examples

- If $n = mk + 1 \geq 2$ then the complete graph K_n is a quasi m-Cayley graph on a cyclic group $\mathbb{Z}_k$.

Examples

- If $n = mk + 1 \geq 2$ then the complete graph K_n is a quasi m-Cayley graph on a cyclic group $\mathbb{Z}_k$.
- Paley graphs $P(q)$ are quasi bicirculants.

The symbol of a quasi m-Cayley graph

The symbol of a quasi m-Cayley graph in the following way:

Let X be a quasi m-Cayley graph on a group G and let $\{U_0, \ldots, U_{m-1}\}$ be the set of m orbits of G on $V(X) - \{\infty\}$. Let $u_i \in U_i$, $i \in \mathbb{Z}_m$, let $S_{i,j}$, $i, j \in \mathbb{Z}_m$ be defined by $S_{i,j} = \{\rho \in G \mid u_i \to \rho(u_j)\}$, and let $S_\infty \subseteq \mathbb{Z}_m$ be defined by

$$S_\infty = \{i \in \mathbb{Z}_m \mid U_i \subseteq N(\infty)\}.$$

Then the family $(S_{i,j})$ together with S_∞ is called the symbol of X relative to $(G; u_0, \ldots, u_{m-1}, S_\infty)$, and determines the adjacencies in the graph. We can assume by renumbering the orbits if necessary that $S_\infty = \{0, \ldots, s-1\}$ for some s.

Quasi-partial difference family

Let G be a group of order n and m, s positive integers with $s \leq m$. A family $\{S_{i,j}\}$ of subsets of G, with $0 \leq i, j \leq m-1$ that satisfy $0 \notin S_{i,i} \forall i$ and $S_{j,i} = -S_{i,j} \forall i, j$ with $i \neq j$, is said to be an (m, n, s, λ, μ) quasi-partial difference family if

$$\sum_{j=1}^{m} |S_{i,j}| = \begin{cases} ns - 1 & \text{if } i \leq s \\ ns & \text{otherwise} \end{cases}, \sum_{j=1}^{s} |S_{i,j}| = \begin{cases} \lambda & \text{if } i \leq s \\ \mu & \text{otherwise} \end{cases} \qquad (1)$$

and if the following identities hold in the group ring $\mathbb{Z}[G]$:

$$\sum_{k=0}^{m-1} S_{i,k} S_{k,j} = \delta_{i,j} \gamma \{0\} + \beta S_{i,j} + \mu' G, \qquad (2)$$

where $\delta_{i,j}$ is the Kronecker delta, $\gamma = ns - \mu$, $\beta = \lambda - \mu$ and

$$\mu' = \begin{cases} \mu - 1 & \text{if } i, j \leq s \\ \mu & \text{otherwise.} \end{cases}$$

Proposition

The quasi m-Cayley graph defined by the symbol $(S_{i,j})$ with $S_\infty = \{0, \ldots, s-1\}$ is an $(mn+1, ns, \lambda, \mu)$-SRG iff $(S_{i,j})$ forms an (m, n, s, λ, μ) quasi-partial difference family.

Example

An example of a $(3,3,1,0,1)$ quasi-partial difference family on the cyclic group C_3:

$S_{0,0} = \emptyset$, $S_{0,1} = \{0\}$, $S_{0,2} = \{0\}$,
$S_{1,1} = \emptyset$, $S_{1,2} = \{1,2\}$, $S_{2,2} = \emptyset$.

Example

An example of a $(3,3,1,0,1)$ quasi-partial difference family on the cyclic group C_3:

$S_{0,0} = \emptyset$, $S_{0,1} = \{0\}$, $S_{0,2} = \{0\}$,
$S_{1,1} = \emptyset$, $S_{1,2} = \{1,2\}$, $S_{2,2} = \emptyset$.

This quasi-partial difference family generates the Petersen graph.

Theorem

Let G be a cyclic group. If $\{S_{i,j}\}$ is a (m,n)-circulant (m,n,s,λ,μ) quasi-partial difference family and if one of the following three conditions is satisfied:

1. n is a prime
2. n is coprime to $(m!)\Delta$
3. $m = 2$ and Δ does not divide $2n$ where $\beta = \lambda - \mu, \gamma = k - \mu$ and $\Delta = \sqrt{\beta^2 + 4\gamma}$

then $\{S_{i,i}\}_{0 \leq i \leq m-1}$ covers all the elements of $G - \{0\}$ the same number of times.

Theorem

If $dm+1$ is a prime power, then a uniform

$$(m, d(md+2), 1, d^2 - md + 3d - 1, d(d+1))$$

QPDF exists.

Theorem

If $dm+1$ is a prime power, then a uniform

$$(m, d(md+2), 1, d^2 - md + 3d - 1, d(d+1))$$

QPDF exists.

An $(m, n, 1, \lambda, \mu)$ QPDF $\{S_{i,j}\}$ on a cyclic group C_n is uniform if the following three conditions hold:

- $\cup_{i=0}^{m-1} S_{i,i} = G - \{0\}$.
- all the cardinalities $|S_{i,i}|$ with $i \geq 1$ are equal.
- all the cardinalities $|S_{i,j}|$ with $i, j \geq 1$ and $i \neq j$ are equal.

Thank you!